Lecture 11: Beginning the Heat Equation

Model Problem: Heat flow in a Metal Rod

- As with modeling for the wave equation, we start with 1 dimension.
- Let $u(t,x)$ denote the temperature at time t of the rod at position x, for $x \in \mathbb{R}$ for now. We focus on two relationships: that between internal energy and external temp, and Fourier's law of heat conduction
- Thermal energy is proportional to a product of density & temperature

$$\underbrace{\mathcal{U}}_{\text{Total Thermal Energy}} = \underbrace{c}_{\text{Specific heat}} \int_a^b \underbrace{\rho}_{\text{Density (we assume constant)}} u \, dx \qquad (A)$$

- Fourier's law describes how heat moves:

$$\underbrace{q}_{\text{Thermal flux}} = -\underbrace{k}_{\text{Thermal conductivity constant}} \frac{\partial u}{\partial x} \qquad (B)$$

- Assume the rod is thermally isolated, so that conservation of energy dictates that energy change = flux difference

$$\frac{d\mathcal{U}}{dt}(t) = q(t,a) - q(t,b) = \int_a^b -\frac{\partial q}{\partial x} \, dx$$

while similarly, differentiating (A) gives

$$\frac{d\mathcal{U}}{dt} = c \int_a^b \rho \frac{\partial u}{\partial t} \, dx$$

Such that

$$\int_a^b c\rho \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

·) as a & b were arbitrary, $c\rho \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$

By B)

$$\frac{\partial u}{\partial t} - \frac{k}{c\rho} \frac{\partial^2 u}{\partial x^2} = 0 \qquad (C)$$

the 1D Heat equation.

· If our rod is of finite length ℓ, we impose B.C.

$$u(t,0) = T_0 \qquad u(t,\ell) = T_1 \qquad \text{(Dirichlet B.C.)}$$

to signify that we have fixed the temperature at the ends (holding in a large bath of water, for example)

These may be reduced to homogeneous (=0) conditions by noting that

$$u_0(x) = T_0\left(1 - \frac{x}{\ell}\right) + T_1 \frac{x}{\ell}$$

Satisfies the B.C. & (C). It is called the equillibrium Solution. By superposition, $u - u_0$ satisfies (C) and has $u(t,0) = u(t,\ell) = 0$.

• Another possible case is having insulated ends, so no heat flows in or out:

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\ell) = 0$$

(Neumann B.C.)

ex.) On the bounded interval $[0, \pi]$, we can find product solutions to the heat equation as in lemma 5.1 of lecture 9. For $u(0) = u(\pi) = 0$, Theorem 5.2 gives helmholtz solutions $\sin(nx)$, so our heat solutions are $u(t,x) = e^{-n^2 t}\sin(nx)$

Product equations:

$$\frac{dV}{dt} = kV$$

$$\frac{d^2\phi}{dx^2} = k\phi$$

Notice $u(t,x) \to 0$ as $t \to \infty$ (losing heat ~~...~~ to approach 0 energy)

If we instead use insulated ends, we get solutions

$$u(t,x) = e^{-n^2 t}\cos(nx)$$

and $n=0$ yields a constant solution. $\square$

- The higher-dimensional heat eqn. may be derived similarly: $q = -k\nabla u$ and local conservation of energy is

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot q = 0$$

giving $\frac{\partial u}{\partial t} - \frac{k}{c\rho}\Delta u = 0$

$\rightsquigarrow$ This can be used to model Brownian Motion, which we will show now

•) Brownian Motion By Einstein's Argument:

- Suppose n particles are distributed on $\mathbb{R}$ and in an interval of time τ, each particle's position changes by a random amount according to a distribution function ϕ.

- The number of particles experiencing a displacement between σ & $\sigma + d\sigma$ is

$$dn = n\,\phi(\sigma)\,d\sigma$$

- Total # of particles is conserved: $\int_{\mathbb{R}} \phi(\sigma)\,d\sigma = 1$

We also assume displacements to be symmetric in distribution:

$$\phi(\sigma) = \phi(-\sigma)$$

- Suppose the distribution of particles at time t is given by $\rho(t,x)$

By our displacement hypothesis,

$$\rho(t+\tau, x) = \int_{-\infty}^{\infty} \rho(t, x-\sigma)\,\phi(\sigma)\,d\sigma \qquad (*)$$

To find an equation for ρ, Einstein takes the Taylor Expansion

$$\rho(t+\tau, x) = \rho(t,x) + \frac{\partial \rho}{\partial t}(t,x)\tau + \frac{1}{2}\frac{\partial^2 \rho}{\partial t^2}(t,x)\tau^2 \ldots$$

and

$$\rho(t, x-\sigma) = \rho(t,x) - \frac{\partial \rho}{\partial x}(t,x)\sigma + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}(t,x)\sigma^2 + \ldots$$

Integrating the second,

$$\int_{-\infty}^{\infty} \rho(t,x-\sigma)\,\phi(\sigma)\,d\sigma = \rho(t,x) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}(t,x)\int_{-\infty}^{\infty}\sigma^2\phi(\sigma)\,d\sigma + \ldots$$

Since $\int_{-\infty}^{\infty} \underbrace{\sigma^{2K+1}\phi(\sigma)}_{\text{odd}}\,d\sigma = 0$ for $K \in \mathbb{N}_0$

Then,

$$\rho(t,x) + \frac{\partial \rho}{\partial t}(t,x)\tau + \ldots = \rho(t,x) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}(t,x)\int_{-\infty}^{\infty}\sigma^2\phi(\sigma)\,d\sigma + \ldots$$

and keeping the leading term gives

$$\frac{\partial \rho}{\partial t}(t,x)\tau = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}(t,x)\int_{-\infty}^{\infty} \sigma^2 \phi(\sigma)d\sigma$$

- From Statistics, we also assume $D = \frac{1}{2\tau}\int_{-\infty}^{\infty} \sigma^2\phi(\sigma)d\sigma$ is constant, ~~...~~. So the equation for ρ becomes

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = 0.$$

$\rightsquigarrow$ Diffusion model of particles.

Scale-Invariant Solution

- Consider the heat equation on $\mathbb{R}$, with physical constant 1

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (H)$$

this will be a case of using "physical symmetries" to guess a solution. We notice that (H) is invariant under a rescaling $(t,x) \to (\lambda^2 t, \lambda x)$ for $\lambda \neq 0$, $\lambda \in \mathbb{R}$.
This suggests a change of variables to the scale-invariant $y = x/\sqrt{t}$ might reduce (H) to an ODE.

- We try to find a solution $q(y) = u(t,x)$ for $t>0$, so the chain rule gives

$$\frac{\partial u}{\partial t} = -\frac{y}{2t}q' \qquad \partial^2 u/\partial x^2 = \frac{1}{t}q''$$

So (H) becomes $q'' = -\frac{y}{2}q'$. Solving by sep. of variables gives $q'(y) = q'(0)e^{-y^2/4}$
and $q(y) = q'(0)\int_0^y e^{-z^2/4}dz + q(0)$, or

$$u(t,x) = C_1\int_0^{x/\sqrt{t}} e^{-y^2/4}dy + C_2$$

- Check that this satisfies (H)!

- To understand the situation $t \to 0$, note

$$\int_0^\infty e^{-s^2/4} ds = \sqrt{\pi} \qquad \text{(the Gaussian integral, proof available upon request)}$$

thus

$$\lim_{t \to 0} u(t,x) = \begin{cases} c_1\sqrt{\pi} + c_2 & x > 0 \\ 0 & x = 0 \\ -c_1\sqrt{\pi} + c_2 & x < 0 \end{cases}$$

Therefore, we pick $c_1 = \frac{1}{\sqrt{4\pi}}$ $c_2 = 1/2$ so

$$\tilde{U}(t,x) = \frac{1}{\sqrt{4\pi}} \int_0^{x/\sqrt{t}} e^{-y^2/4} dy + 1/2$$

has $\lim_{t \to 0} \tilde{U}(t,x) = \begin{cases} 1 & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases} = \Theta(x)$ "Heaviside Step Function"

- Why? Behind the scenes, we're using more complex theory to arrive at ~~...~~ convolutions. If we want initial condition $u(0,x) = \varphi(x) \in C_c^\infty(\mathbb{R})$, we observe

$$\int_{-\infty}^{\infty} \varphi'(z)\Theta(x-z)dz = \int_{-\infty}^{x} \varphi'(z) dz = \varphi(x)$$

So that we attempt to set

$$u(t,x) = \int_{-\infty}^{\infty} \varphi'(z)\tilde{U}(t,x-z)dz$$

Since $\tilde{u}$ is C^1 in $t>0$,

$$u(t,x) = -\int_{-\infty}^{\infty} \varphi(z) \frac{\partial \tilde{U}}{\partial z}(t,x-z)\, dz$$

$$= +\int_{-\infty}^{\infty} \frac{\varphi(z)}{\sqrt{4\pi t}} e^{-(x-z)^2/4t} dz$$

Set $H_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ and

$$u(t,x) = \underbrace{\int_{-\infty}^{\infty} H_t(x-z)\varphi(z)dz}_{\text{Convolution}}$$

- We check this solution in the next lecture.